For the special case $e = 0$ and $\bar{\bar{a}}_i = 0$, it is possible to derive a first integral of these equations. This can be done by multiplying Eqs. (24–26) by u', v', w', respectively, and summing. We obtain

$$[(u')^2 + (v')^2 + (w')^2]' = (3v^2 - w^2)' - (A/\sigma^3)(\sigma^2)' \tag{27}$$

Integrating this equation (with respect to θ) we then obtain

$$(u')^2 + (v')^2 + (w')^2 - 3v^2 + w^2 - 2A/\sigma = \text{constant} \tag{28}$$

This first integral can be considered as an "extension" of the Jacobi first integral[7] for the restricted three-body problem in two dimensions.

Finally we would like to consider the two-dimensional restricted three-body problem (i.e., the spacecraft trajectory is in the $x - y$ plane) with $e = 0$ and $a_i = 0$.

To treat this problem, we introduce

$$u = \sigma \cos\phi, \qquad v = \sigma \sin\phi \tag{29}$$

Equation (28) then becomes

$$(\sigma')^2 + \sigma^2(\phi')^2 - 2A/\sigma = 3\sigma^2 \sin^2\phi \tag{30}$$

Furthermore, by differentiating $\sigma^2 = u^2 + v^2$ twice and using Eqs. (24–26) and (28) we obtain

$$\sigma\sigma'' + (\sigma')^2 - 2\sigma^2\phi' - A/\sigma = 6\sigma^2 \sin^2\phi \tag{31}$$

It is easy to verify that (as in the classical case) the system (30), (31) has an (implicit) solution when $\phi(\theta)$ = constant. Moreover we can reduce Eqs. (30) and (31) to a first-order system by changing the independent variable from θ to σ and introduce $p = \mathrm{d}\sigma/\mathrm{d}\theta$ as a new dependent variable. After some algebra we obtain

$$p^2\left[1 + \sigma^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2\right] - \frac{2A}{\sigma} = 3\sigma^2 \sin^2\phi \tag{32}$$

$$p\left(\sigma\frac{\mathrm{d}p}{\mathrm{d}\sigma} - p\right) = 2\sigma^2 p\left[p\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2 + \frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right] - \frac{3A}{\sigma} \tag{33}$$

Conclusions

The equations derived in this Note for the restricted three-body problem in three dimensions closely resemble those that were derived by Carter and the author in a recent paper. They will be amenable to treatment by analytic and computational techniques. In this Note we considered only the equations of motion in a gravitational force field. However, we would like to conjecture that similar equations can be derived for the same system in a general central force field with the presence of linear drag as was done in the case of a spacecraft and a satellite.

As to the accuracy of the approximate rendezvous equations, we observe that we made two approximations in their derivation. The first results from neglecting the spacecraft's influence on the motion of the other two bodies in the system. The second is from the linearization of the equations of motion under the assumption $r \ll R$.

From a practical point of view, the first approximation is well suited for all celestial applications. The second becomes more accurate as $r/R \to 0$. Thus, unless the two celestial bodies are extremely close to each other the linearization of the equations of motion is well justified.

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Energy-Based Stabilization of Angular Velocity of Rigid Body in Failure Configuration

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I. Introduction

THE problem of stabilization of the angular velocity of a rigid body, modeling a simple satellite, has been addressed by several authors and is discussed and solved in this Note from a new perspective. From a control theoretic point of view the most interesting and studied case is the one of a body operating in failure configuration, that is, with only one or two independent actuators acting on the system. More precisely, in Refs. 1–4 it was shown that the zero solution of Euler's angular velocity equations can be made asymptotically stable by means of two control torques, whereas in Refs. 5–8 the same problem has been addressed and solved in the case of only one control torque. Robust stabilization has been studied in Refs. 9 and 10, and stabilization using partial state information has been addressed in Ref. 11. Finally, the problem of stabilization of nonzero (relative) equilibria has been studied in Refs. 11–13. In almost all of the aforementioned papers (Refs. 3, 12, and 13 are notable exceptions) the stabilization problem has been addressed and solved without making use of the special structure of the system to be controlled, that is, stabilizing control laws have been derived using various techniques, such as backstepping, center manifold theory, homogeneity considerations, and control Lyapunov functions

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methods, but without exploiting the physical and mathematical properties of the system. On the contrary, in the present Note the Hamiltonian structure of the system is exploited in the design of stabilizing control laws. New state feedback control laws, asymptotically stabilizing the zero equilibrium of the underactuated Euler's equations, are proposed. Note that the design procedures proposed in this Note not only make use of the Hamiltonian structure of the system to be controlled, but yield closed-loop systems possessing a Hamiltonian structure.

There are various advantages in preserving (in the closed loop) the physical (Hamiltonian) structure of the system (see Ref. 14 for further details). First, the control action has a clear physical interpretation as the interconnection of the system with the controller. Hence, stabilization can be understood in terms of energy balance between system and controller. Second, because Hamiltonian systems are passive with respect to physically meaningful outputs, a margin of robustness, vis a vis uncertain parameters and unmodeled dynamics, is ensured. Finally, the control parameters can be interpreted as playing the role of dampers and springs.

II. Model

Consider a rigid body in an inertial reference frame, and let x_1, x_2, and x_3 denote the components of the angular momentum vector along a body-fixed reference frame having the origin at the center of gravity and consisting of the three principal axes. The Euler's equations for the rigid body subject to m external control torques, generated by pairs of gas jet actuators, are

$$\dot{x}_1 = (1/I_3 - 1/I_2)x_2x_3 + B_1u$$

$$\dot{x}_2 = (1/I_1 - 1/I_3)x_3x_1 + B_2u$$

$$\dot{x}_3 = (1/I_2 - 1/I_1)x_1x_2 + B_3u \tag{1}$$

where $I_1 > 0$, $I_2 > 0$, and $I_3 > 0$ denote the principal moments of inertia, $u \in \mathbb{R}^m$ the control torques, and $B_i^T \in \mathbb{R}^m$ the coefficients depending on the locations of the actuators. (Note that gas jet actuators are operated in an on–off mode; hence, they generate a discontinuous control action. In what follows we disregard this important issue and concentrate on the design of continuous control laws.) System (1) can be rewritten as

$$\dot{x} = [J(x) - R(x)]\frac{\partial H^T}{\partial x} + Bu$$

with $B^T = [B_1^T, B_2^T, B_3^T]$, that is, as a port-controlled Hamiltonian system with internal Hamiltonian (energy) $H(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$, interconnection (Poisson structure)

$$J(x) = -J^T(x) = \begin{bmatrix} 0 & x_3/I_3 & -x_2/I_2 \\ -x_3/I_3 & 0 & x_1/I_1 \\ x_2/I_2 & -x_1/I_1 & 0 \end{bmatrix}$$

and internal dissipation $R(x) = R^T(x) = 0$. Note that, along the trajectories of the uncontrolled system, one has $\dot{H} = 0$, that is, the function $H(x)$ is a Lyapunov function for the zero equilibrium, and it is conserved along the motion of the system with $u = 0$.

III. Structure Preserving Stabilization

In this section we briefly discuss two design methods, namely, the methods of structure preserving stabilization and of asymptotic structure preserving stabilization, for general port-controlled Hamiltonian systems.

The problem of structure preserving stabilization, or, equivalently, of interconnection and damping assignment, for a general port-controlled Hamiltonian system can be posed as follows.

A. Structure Preserving Stabilization Problem

Given the (port-controlled) Hamiltonian system

$$\dot{x} = [J(x) - R(x)]\frac{\partial H^T}{\partial x} + B(x)u \tag{2}$$

with state x, control u, internal Hamiltonian $H(x)$, interconnection $J(x) = -J^T(x)$, dissipation $R(x) = R^T(x)$, input matrix $B(x)$, and a point x_e in the state space, find (if possible) a modified Hamiltonian function $H_a(x)$, a modified interconnection $J_a(x) = -J_a^T(x)$, a modified dissipation $R_a(x) = R_a^T(x)$, and a (state feedback) control law $u = u(x)$, such that the closed-loop system is a Hamiltonian system with modified energy, interconnection, and damping, that is,

$$\begin{aligned}\dot{x} &= [J(x) - R(x)]\frac{\partial H^T}{\partial x} + B(x)u(x) \\ &= \{J(x) + J_a(x) - [R(x) + R_a(x)]\}\frac{\partial (H + H_a)^T}{\partial x}\end{aligned}$$

and the point x_e is a locally (globally) asymptotically stable equilibrium of the closed-loop system.

The first requirement can be recast in terms of a system of partial differential equations (PDEs) in the (unknown) functions $H_a(x)$ and $u(x)$, once candidate modifications of the interconnection $J_a(x)$ and of the dissipation $R_a(x)$ have been selected. The solvability of this system of PDEs has been studied in detail in Ref. 15, where a set of sufficient conditions for solvability has been proposed. Note that to design a stabilizing control law $u = u(x)$ it is not necessary to solve explicitly the system of PDEs, but it is enough to guarantee the existence of a solution with some special boundary conditions.

The second requirement is achieved if

$$\left.\frac{\partial (H + H_a)}{\partial x}\right|_{x = x_e} = 0$$

that is, if x_e is an equilibrium of the closed-loop system, if x_e is a strict local (global) minimum of $H(x) + H_a(x)$ and $R(x) + R_a(x) \geq 0$, that is, the closed-loop system is Lyapunov stable, and if the trajectories of the closed-loop system converge asymptotically to x_e.

Note that a basic ingredient in the preceding problem formulation is that the state spaces of the system to be controlled and of the target closed-loop system have the same dimension. This constraint can be removed at the expenses of a weaker design goal, as explained hereafter.

B. Asymptotic Structure Preserving Stabilization Problem

Given the port-controlled Hamiltonian system (2) with state x and control u, and the target autonomous port-controlled Hamiltonian system with dissipation

$$\dot{\xi} = [j(\xi) - r(\xi)]\frac{\partial V^T}{\partial \xi} \tag{3}$$

with state ξ, internal Hamiltonian $V(\xi)$, interconnection $j(\xi) = -j^T(\xi)$, and dissipation $r(\xi) = r^T(\xi) \geq 0$, find (if possible) a mapping $z = \phi(x)$ and a control law $u = u(x)$ such that the resulting closed-loop system has the following properties:

1) The set $z = 0$ is invariant and attractive.

2) The (well-defined) restriction of the closed-loop system to the set $z = 0$ is a diffeomorphic copy of the target port-controlled Hamiltonian system (3).

It is obvious that the asymptotic structure preserving stabilization problem is simpler to solve than the structure preserving stabilization one. Nevertheless, in both cases the solvability of the problem relies on the solvability of a system of PDEs with special boundary conditions.

Remark 1: From the preceding definition it is apparent that the structure preserving stabilization problem is a special case of the asymptotic structure preserving stabilization problem. Moreover, the latter takes its name from the fact that the preservation of the structure is an asymptotic property, that is, the trajectories of the closed-loop system converge toward the trajectories of the target Hamiltonian system as the time goes to infinity.

IV. Stabilization of the Zero Equilibrium

We now discuss the structure preserving stabilization problem for the zero equilibrium. Note that the solvability of the problem

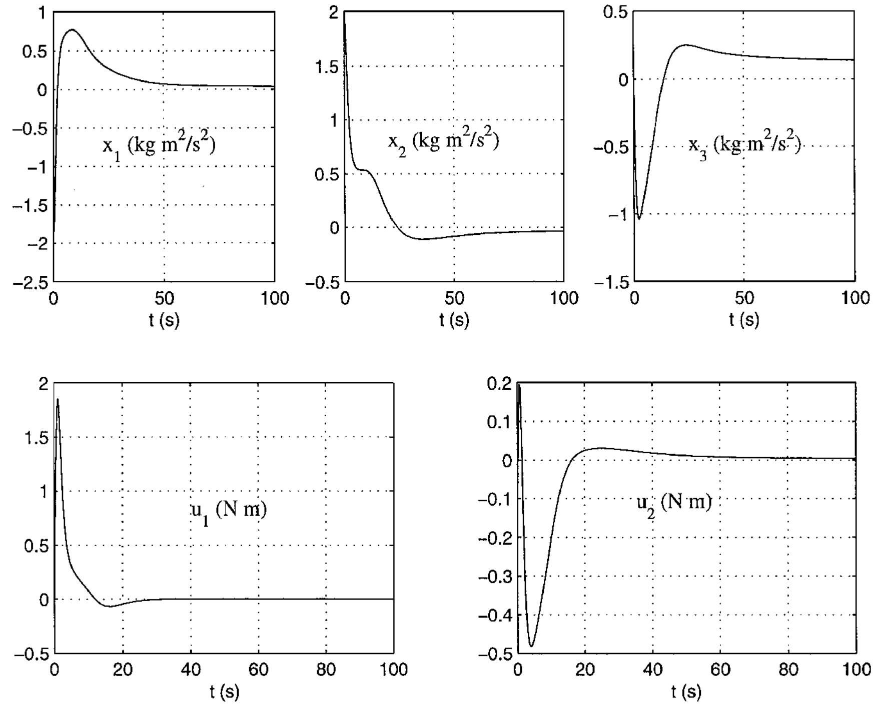

Fig. 1 State and control histories for the control law achieving structure preserving stabilization of the zero equilibrium.

and the complexity of the solution depend on the structure of the input matrix B. In particular, the problem is trivial if the input matrix is constant and has a nonzero element in any row. In fact, if this is the case, the control law $u = -B^T x$, that is, a simple damping injection, achieves the goal with $H_a(x) = 0$, $J_a(x) = 0$, and $R(x) = R = BB^T \geq 0$. However, if the preceding conditions are not satisfied the problem is much more involved, as discussed in the following statement.

Proposition 1: Consider system (1) with $B_1 = [1, 0]$, $B_2 = [0, 1]$, $B_3 = [0, 0]$, and the point $x_0 = (0, 0, 0)$. Assume that $I_1 \neq I_2$. Then the structure preserving stabilization problem is (globally) solvable and a solution (the matrix $J_a(x)$ is skew symmetric; hence, the upper diagonal terms have not been specified) is

$$H_a(x) = -A_3x_1x_3 + \tfrac{1}{2}A_3^2x_3^2 + x_3^2x_2 + \tfrac{1}{2}x_3^4$$

$$R_a(x) = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 + 4A_3^2x_3^4 & -2A_3^2x_3^3 \\ 0 & -2A_3^2x_3^3 & A_3^2x_3^2 \end{bmatrix}$$

$$J_a(x) = \begin{bmatrix} 0 & \star & \star \\ x_3/I_3 - \left(A_3x_1 - 2A_3^2x_3^3 - 2x_3x_2 - 2x_3^3\right)A_3 & 0 & \star \\ -A_3x_3^2\left(A_3^2 + 1\right) - x_2/I_2 & -x_1/I_2 & 0 \end{bmatrix}$$

$$u_1(x) = A_3x_3^3 - 2A_3x_3x_2^2 - 2A_3x_2x_3^3 + A_3^2x_2x_1 + 2A_3^3x_3^3 - A_3^4x_3^2x_1 + A_3^5x_3^3 - (1/I_3 - 1/I_2)x_2x_3 - R_1(x_1 - A_3x_3)$$

$$u_2(x) = -2A_3^2x_3^2x_2 - (1/I_1 - 1/I_3)x_1x_3 - A_3x_1x_3 - R_2\left(x_2 + x_3^2\right) \tag{4}$$

with $A_3 = 1/I_2 - 1/I_1$ and R_i positive constants.

Proof: To begin, note that the function

$$H(x) + H_a(x) = \tfrac{1}{2}(x_1 - A_3x_3)^2 + \tfrac{1}{2}\left(x_2 + x_3^2\right)^2 + \tfrac{1}{2}x_3^2 \tag{5}$$

is positive definite and proper and that the matrix R_a is positive definite for any nonzero x_3 and positive semidefinite for $x_3 = 0$. Moreover, along the trajectories of the closed-loop system, one has

$$\dot{H} + \dot{H}_a = -\frac{\partial[H(x) + H_a(x)]}{\partial x} R_a(x) \left[\frac{\partial[H(x) + H_a(x)]}{\partial x}\right]^T \leq 0$$

which proves Lyapunov stability of the closed-loop system. Asymptotic stability is then proved using a standard La Salle argument and the properties of the matrix $R(x)$ discussed earlier. □

Remark 2: Note that a substantial change in the interconnection has been introduced and that the damping matrix $R_a(x)$ has nonzero cross terms, that is, it is not a diagonal matrix.

Remark 3: The energy function (5), which is a control Lyapunov function for the system, can be also derived using backstepping arguments (see Ref. 16 for further details). However, the control laws resulting from a straightforward application of a backstepping procedure do not possess, in general, the property discussed earlier (i.e., they do not preserve the structure of the system), or, similarly, the closed-loop system cannot be written as a port-controlled Hamiltonian system with dissipation.

Simulations have been carried out and sample state and control histories are displayed in Fig. 1. Note the nonexponential convergence rate.

V. Asymptotic Structure Preserving Stabilization

In the preceding section we have seen that the problem of structure preserving stabilization for the zero equilibrium of system (1) is solvable. However, the solution is very complex: The closed-loop system is a port-controlled Hamiltonian system with an interconnection that is completely different from the interconnection of the uncontrolled system, and the control law is composed of several terms. Simpler control laws can be obtained if the less ambitious problem of asymptotic structure preserving stabilization is posed.

Proposition 2: Consider system (1) with $B_1 = [1, 0]$, $B_2 = [0, 1]$, $B_3 = [0, 0]$, $I_1 \neq I_2$, and the target port-controlled Hamiltonian system $\dot{\xi} = -\alpha\xi^3$, with $\xi \in \mathbb{R}$ and $\alpha > 0$, that is, $j(\xi) = 0$, $r(\xi) = \alpha\xi^2$, and $V(\xi) = \frac{1}{2}\xi^2$.

Then the asymptotic structure preserving stabilization problem is (globally) solvable, and a solution is

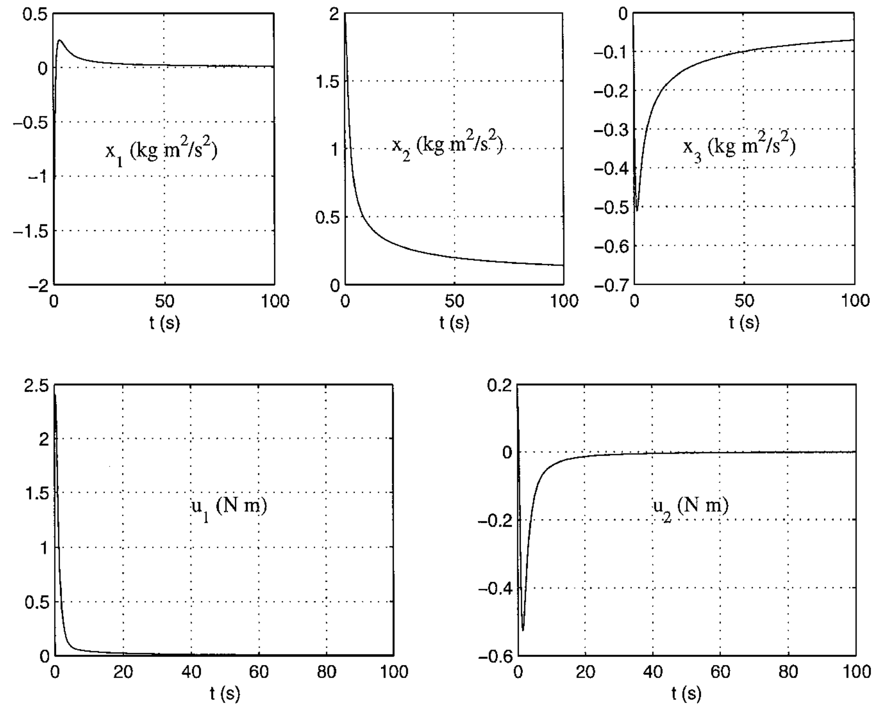

Fig. 2 State and control histories for the control law achieving asymptotic structure preserving stabilization of the zero equilibrium.

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + \lambda_1 x_3^2 \\ x_2 + \lambda_2 x_3 \end{bmatrix} \tag{6}$$

$$u_1(x) = -A_1 x_2 x_3 - 2\lambda_1 A_3 x_1 x_2 x_3 - k_1\left(x_1 + \lambda_1 x_3^2\right)$$
$$u_2(x) = -A_2 x_1 x_3 - \lambda_2 A_3 x_1 x_2 - k_2(x_2 + \lambda_2 x_3) \tag{7}$$

with $\lambda_1\lambda_2 A_3 = -\alpha$, $k_1 > 0$, $k_2 > 0$, $A_1 = 1/I_3 - 1/I_2$, $A_2 = 1/I_1 - 1/I_3$, and $A_3 = 1/I_2 - 1/I_1$.

Proof: Consider the closed-loop system (1–7). Simple calculations show that

$$\dot{z}_1 = -k_1 z_1, \qquad \dot{z}_2 = -k_2 z_2$$

that is, the set $z = 0$ is invariant and attractive. Moreover, the well-defined restriction of the closed-loop system (1–7) to the set $z = 0$ is

$$\dot{x}_3 = A_3(x_1 x_2)_{z=0} = A_3\lambda_1\lambda_2 x_3^3 = -\alpha x_3^3$$

hence the claim. □

Remark 4: The selection of $\dot{\xi} = -\alpha\xi^3$ as target Hamiltonian system is related to the fact that the system (1) cannot be exponentially stabilized by any continuous control law because, under the assumption of Proposition 2, the linearized system has one uncontrollable eigenvalue on the imaginary axis. However, if non-differentiable control laws are considered, it is possible to prescribe faster target dynamics, for example, $\dot{\xi} = -|\xi|^{\beta}\xi$ with $\beta > 0$.

Remark 5: Note that the control law (7) is much simpler than the control law (4), yet it provides asymptotic stability and is such that the closed-loop system behaves like the target port-controlled Hamiltonian system asymptotically.

Simulations have been carried out and state and control histories, with the same initial conditions used for the simulations in Fig. 1, are displayed in Fig. 2. Note the nonexponential convergence rate.

VI. Conclusions

The stabilization problem for the angular velocity of a rigid body around the zero equilibrium has been studied from a new point of view. In particular, it is shown that it is possible to achieve asymptotic stability using control laws that admit an interpretation in terms of energy flow and dissipation. Further work is in progress to obtain stabilizing control laws for the complete model, that is, kinematic and dynamic, of a simple satellite.

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